# Multisine Multiexcitation in Frequency Response Function Estimation

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## Introduction

TRUE random single excitation has been widely used in frequency response function (FRF) estimation. It reduces test times when compared to swept or stepped sinusoidal excitation. Transient excitation presents the same advantage, but it has handicaps related to digital signal processing difficulties and difficulties in supplying enough energy in a short period of time without causing nonlinear behavior due to high force amplitudes.

When true random excitation is used, averaging is mandatory due to the stochastic character of the signals. Furthermore, some sort of windowing is necessary to reduce leakage in the discrete Fourier transform (DFT). In vibration analysis problems where stochastic phenomena are present, random signal processing techniques are unavoidable. However, this is usually not the case in modal testing, where multifrequency excitation, rather than random excitation, is desired. For this purpose, it is much more reasonable to synthesize an excitation signal by adding sinusoids of arbitrary amplitudes.

Synthesized excitation signals may be generated numerically and then sent to a digital-to-analog converter, which produces an analog signal to drive a shaker. This signal may be made periodic with a period equal to the observation window, so that its DFT is leakage-free. It can also be made transient, in which case it is desirable that both the excitation and the response signals vanish within the observation window. Such signals have been called pseudorandom, periodic random, or chirp in the former case, and burst random or burst chirp in the latter.

More recently, a generalized approach was given to this problem[1] after the work of Schroeder,[2] which was first used in mechanical system identification by Burrows.[3] True random signals have nonetheless continued to be used in FRF estimation in the case of simultaneous broadband multiexcitation.[4] Otherwise, in modern stepped sine multiexcitation methods,[5] the phase is usually random. It will be shown in this paper that the same results can be obtained using Schroeder-phased multisine excitation signals with linearly independent sets of constant phases.

## Multisine Excitation Signals

Given a linear time-invariant structure, the dynamic response at the $i$th degree of freedom (DOF) to a force input at the $j$th DOF is given in the frequency domain by an FRF, say, $H_{ij}(f)$. If the input signal, $p_j(t)$, is periodic with period $T$, it can be expanded in a Fourier series with complex coefficients $P_j(f_k)$, where $f_k = k/T$, $k$ = integer. So can the output signal, $x_i(t)$, with coefficients $X_i(f_k)$. The input/output relation is given by

$$X_i(f_k) = H_{ij}(f_k)P_j(f_k) \tag{1}$$

If $P_j(f_k) \neq 0$ for $k_1 \leq k \leq k_2$, $H_{ij}(f_k)$ can be determined in the frequency range $f_{k_1} \leq f_k \leq f_{k_2}$ with frequency resolution $1/T$. Such a periodic excitation signal is called a "multisine signal."

Complex Fourier series can be calculated by the DFT with practically no error, provided that the sampling frequency is high enough to prevent aliasing and that an integer number of periods of the signal is acquired in the observation time $T$. A multisine signal may be easily synthesized with

$$p_j(t) = \sum_{k=k_1}^{k_2} A_k \cos(2\pi kt/T + \phi_k) \tag{2}$$

For the usual case of a flat spectrum with $A_k = A$ = const, using Eq. (2) with $\phi_k = \phi$ = const, the signal that is obtained is a sort of periodic impact. This signal has one of the disadvantages of transient excitation signals, namely, high amplitudes. Schroeder[3] addressed this problem and searched for the minimization of the crest factor of multisine signals. The crest factor is defined as the ratio between the highest amplitude of the signal and its rms value. The minimization is made for a given amplitude distribution by a proper choice of the phase angles $\phi_k$.

For an arbitrarily shaped amplitude spectrum, Schroeder[2,3] derived a formula that can be written as

$$\phi_k = 2\pi \sum_{l=k_1}^{k} \frac{A_l^2}{\sum_{i=k_1}^{k_2} A_i^2} \tag{3}$$

which, for a flat spectrum, $A_l = A$ = const, can be simplified to

$$\phi_k = 2\pi \left[ \frac{k^2 + k - k_1^2 + k_1}{2(k_2 - k_1 + 1)} \right] \tag{4}$$

Schroeder gives the result for $k_1 = 1$, $k_2 = N$, which is[2]

$$\phi_k = \frac{\pi k^2}{N} \tag{5}$$

If the phase angle is restricted to 0 and $\pi$, Eq. (2) may be written, for $k_1 = 1$ and $k_2 = N$,

$$P_j(t) = \sum_{k=k_1}^{N} A_k \cos\left(2\pi \frac{kt}{T}\right) \tag{6}$$

where, for the flat spectrum case,

$$A_k = A\left\{1 - 2\left[\frac{k^2}{2N}\right]_{\text{mod}2}\right\} \tag{7}$$

In Eq. (7), $A$ is the constant amplitude, and

$$\left[\frac{k^2}{2N}\right]_{\text{mod}2} \begin{cases} = 1 & \text{if integer } \left(\dfrac{k^2}{2N}\right) \text{ is odd} \\ = 0 & \text{if integer } \left(\dfrac{k^2}{2N}\right) \text{ is even} \end{cases} \tag{8}$$

With multisine excitation, the FRF may be obtained from Eq. (1) using a single period of the input and output signals. Although not mandatory, averaging can reduce additive random measurement noise on the estimated FRF. The number of averages in this case may be considerably smaller than in the case of true random excitation.

## Multisine Multiexcitation Scheme

Consider simultaneous multiexcitation with $n_e$ external forces. Numbering the structural DOFs so that the first $n_e$ DOFs correspond to the excitation locations, it is possible to write, without loss of generality, that the response at DOF $i$ is given by

$$X_i(f_k) = \sum_{j=1}^{n_e} H_{ij}(f_k)P_j(f_k) \tag{9}$$

Equation (9) can be arranged in matrix form for each excitation frequency $f_k$:

$$[P_1(f_k)P_2(f_k)\cdots P_{n_e}(f_k)] \begin{Bmatrix} H_{i1}(f_k) \\ H_{i1}(f_k) \\ \vdots \\ H_{in_e}(f_k) \end{Bmatrix} = X_i(f_k) \tag{10}$$

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A system of $n_s$ equations like Eq. (10), formed by applying different sets of simultaneous multisine excitation signals to the structure being tested, can be put in the following form:

$$P_k H_i(f_k) = X_i(f_k) \tag{11}$$

If $n_s \geq n_e$ and if the amplitude distributions among the shakers for the different sets are not linearly dependent, Eq. (11) can be solved for $H_i(f_k)$. For each different frequency $f_k$, an $(n_s \times n_e)$ system of equations has to be solved. The matrix $P_k$ to be inverted is the same for all the response stations. Hence, it is possible to write

$$P_k H_k = X_k \tag{12}$$

where

$$H_k = [H_1(f_k) H_2(f_k) \cdots H_{n_r}(f_k)] \tag{13}$$

$$X_k = [X_1(f_k) X_2(f_k) \cdots X_{n_r}(f_k)] \tag{14}$$

The solution, $\hat{H}_k$ may be calculated in the least-squares sense:

$$\hat{H}_k = P_k^+ X_k \tag{15}$$

where $P_k^+$ is the pseudoinverse of $P_k$.

The $n_s$ linearly independent sets of $n_e$ force amplitudes each may be obtained by multiplying $n_e$ Schroeder-phased multisine signals by $n_s$ orthogonal vectors, which can be easily created using the Gram-Schmidt orthogonalization procedure. It is interesting to note that the particular case where $P_k$ is the identity matrix corresponds to the single excitation case.

When a constant-amplitude multisine signal is used to drive a shaker, the dynamics of the coupled system composed of the shaker, the structure, and the other shakers modifies the amplitude spectrum of the resultant force exerted by the shaker on the structure. If a constant amplitude excitation spectrum is desired, the resultant distorted force spectrum can be measured, and the proper amplitude correction can easily be made. Also, since the various shakers operate simultaneously, the vibrations at a shaker location caused by another shaker at a different location may produce residual forces due to the impedance of the shaker moving table and attachment assembly. Nevertheless, starting with $n_s$ sets of $n_e$ orthogonal force amplitudes, the chance of having a rank deficient matrix $P_k$ is very small. Moreover, if $P_k$ eventually becomes ill-conditioned for a given frequency $f_k$, the amplitude and phase distribution of the forces can be easily changed.

The proposed Schroeder-phased multisine multiexcitation test consists of the following steps:

1) Build a multisine signal with a suitable amplitude distribution in the frequency range of the test and with Schroeder's phase distribution [Eq. (4)].

2) Generate a set of $n_s$ orthogonal vectors of $n_e$ elements each using a random number generator and Gram-Schmidt orthogonalization.

3) Build a set of $n_e$ multisine signals by multiplying the multisine signal by the $n_e$ components of one of the random orthogonal vectors.

4) Apply the multisine signals simultaneously to the structure and cycle it a few times in order to reach the steady-state condition. Calculate the DFTs of the input and output signals, averaging if necessary to reduce additive measurement random noise.

5) Repeat steps 3 and 4 $n_s$ times ($n_s \geq n_e$), each time using a different orthogonal vector.

6) Build matrix $P_k$ for each frequency $f_k$ and calculate $\hat{H}_k$ using Eq. (15).

## Conclusions

A technique for FRF estimation using multisine multiexcitation was presented. It was shown that the versatility of multisine signals, which allow leakage-free DFTs, can be extended to simultaneous multiexcitation tests. It is hoped that this technique will help improving FRF estimation in modal testing applications.

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## References

[1]Schoukens, J., Pintelon, R., Ouderaa, E. V. D., and Rennegoog, J., "Survey of Excitation Signals for FFT Based Signal Analyzers," *IEEE Transactions on Instrumentation and Measurement,* Vol. 37, No. 3, 1988, pp. 342–352.

[2]Schroeder, M. R., "Synthesis of Low-Peak-Factor Signals and Binary Sequences With Low Autocorrelation," *IEEE Transactions on Information Theory,* Jan. 1970, pp. 85–89.

[3]Burrows, C. R., "An Appraisal of Schroeder-Phased Harmonic Signals for Bearing Identification," American Society of Mechanical Engineers, New York, paper 80-WA/DSC-34, 1980.

[4]Leuridan, J., De Vis, D., Auweraer, H. V. D., and Lembregts, F., "A Comparison of Some Frequency Response Measurement Techniques," *Proceedings of the 4th International Modal Analysis Conference,* Los Angeles, CA, Jan. 1985, pp. 908–918.

[5]Hunt, D. L., Vold, H., and Williams, R., "Modal Testing Using Modern Sine Excitation," *Proceedings of the 7th International Modal Analysis Conference,* Las Vegas, NV, Feb. 1989, pp. 1209–1213.